Module-1: Residue Theorem

1 Introduction

Let $f(z)$ be analytic at a point $z = z_0$. Then $f(z)$ is analytic in some neighbourhood $N_\delta(z_0)$ of z_0. If C is a positively oriented closed rectifiable curve contained in $N_\delta(z_0)$, then by Cauchy fundamental theorem we have $\int_C f(z)dz = 0$. However, if $f(z)$ is not analytic at finitely many isolated singularities inside C, then the above argument fails. This means that each of these singularities contributes a specified value to the value of the integral. This motivates to generalize Cauchy fundamental theorem to functions which have isolated singularities. This generalization results in the Residue theorem. This result is one of the most important and often used, tools that applied scientists need, from the theory of complex functions.

Residue at a Finite Point

Suppose the function $f(z)$ has an isolated singularity at $z = z_0$ $(\neq \infty)$. Then in some deleted neighbourhood of z_0 $(0 <\mid z - z_0 \mid< \delta)$, $f(z)$ can be represented by Laurent's series of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \ \ldots \ .$$

Then, the coefficient of $1/(z - z_0)$ i.e. b_1 is called the residue of $f(z)$ at z_0 and is denoted by $Res\ (f; z_0)$. Thus $b_1 = \frac{1}{2\pi i} \int_C f(z)dz$ where C is any positively oriented simple closed rectifiable curve enclosing z_0 and contained in the neighbourhood.

The residue at the isolated singularity $z = z_0$ $(\neq \infty)$ may also be defined as the integral

$$\frac{1}{2\pi i} \int_C f(z)dz$$

where C is any positively oriented simple closed rectifiable curve lying in the domain $0 <\mid z - z_0 \mid< \delta$ which enclose z_0 but no other singularity of $f(z)$.

The following theorem shows how can we calculate residue of a function which has a pole of order m at the point $z = z_0$.

Theorem 1. *If $f(z)$ has a pole of order m at $z = z_0$, then*

$$Res\ (f; z_0) \ = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z)].$$

Proof. Since z_0 is a pole of f of order m, in some deleted neighbourhood of z_0 we can write

$$f(z) \ = \phi(z) + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \ . \ . \ . \ + \frac{b_m}{(z - z_0)^m}, \tag{1}$$

where $\phi(z)$ is analytic at $z = z_0$ and $b_m \neq 0$. From (1) we get

$$\begin{aligned} (z - z_0)^m f(z) \ &= (z - z_0)^m \phi(z) + b_1 (z - z_0)^{m-1} + b_2 (z - z_0)^{m-2} + \ . \ . \ . \ + b_m. \\ Thus, \quad \frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z)] \ &= \frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m \phi(z)] + b_1 \cdot (m-1)!. \end{aligned}$$

Since $\lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}}(z - z_0)^m \phi(z) \ = 0$, we obtain from above

$$Res\ (f; z_0) \ = b_1 \ = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z)].$$

This proves the result. □

Note 1. *If z_0 is a simple pole of f, then*

$$Res\ (f; z_0) \ = \lim_{z \to z_0} (z - z_0) f(z).$$

Theorem 2. *Let $f(z)$ be analytic at z_0 with $f(z_0) \neq 0$ and $g(z)$ has a simple zero at z_0. Then*

$$Res\ \left(\frac{f(z)}{g(z)}; z_0 \right) \ = \frac{f(z_0)}{g'(z_0)}.$$

Proof. We put $F(z) \ = \frac{f(z)}{g(z)}$. Since z_0 is a simple pole of $F(z)$, we have

$$\begin{aligned} Res\ (F; z_0) \ &= \ \lim_{z \to z_0} (z - z_0) F(z) \ = \lim_{z \to z_0} (z - z_0) \frac{f(z)}{g(z)} \\ &= \ \lim_{z \to z_0} \frac{f(z)}{\frac{g(z) - g(z_0)}{z - z_0}} \ = \frac{f(z_0)}{g'(z_0)}. \end{aligned}$$

This proves the result. □

Theorem 3. *Let $f(z)$ be analytic at z_0 such that $f(z_0) \neq 0$ and $g(z)$ has a zero of order two at z_0. Then*

$$Res\left(\frac{f(z)}{g(z)}; z_0\right) = \frac{6f'(z_0)g''(z_0) - 2f(z_0)g'''(z_0)}{3(g''(z_0))^2}. \tag{2}$$

Proof. Since the function $g(z)$ has a zero of order two at z_0, we have

$$g(z) = (z - z_0)^2 \nu(z),$$

where $\nu(z)$ is analytic at z_0 and $\nu(z_0) \neq 0$. Thus

$$\frac{f(z)}{g(z)} = \frac{f(z)/\nu(z)}{(z - z_0)^2}$$

has a pole of order 2, and its residue at z_0 is given by

$$\begin{aligned} Res\left(\frac{f(z)}{g(z)}; z_0\right) &= \lim_{z\to z_0} \frac{d}{dz}\left[(z - z_0)^2 \frac{f(z)}{g(z)}\right] \\ &= \lim_{z\to z_0} \frac{d}{dz}\left[\frac{f(z)}{\nu(z)}\right] \\ &= \lim_{z\to z_0} \frac{f'(z)\nu(z) - f(z)\nu'(z)}{(\nu(z))^2} \\ &= \frac{f'(z_0)\nu(z_0) - f(z_0)\nu'(z_0)}{(\nu(z_0))^2}. \end{aligned} \tag{3}$$

Now from the relation $g(z) = (z - z_0)^2\nu(z)$, we obtain

$$\begin{aligned} g'(z) &= 2(z - z_0)\nu(z) + (z - z_0)^2\nu'(z); \\ g''(z) &= 2\nu(z) + 4(z - z_0)\nu'(z) + (z - z_0)^2\nu''(z); \\ g'''(z) &= 6\nu'(z) + 6(z - z_0)\nu''(z) + (z - z_0)^2\nu'''(z). \end{aligned}$$

These relations imply $\nu(z_0) = g''(z_0)/2$ and $\nu'(z_0) = g'''(z_0)/6$. Incorporating these values in (3) we can get the required result. □

Example 1. *Find the residues of the function $f(z) = \frac{1}{(z^3-1)(z+1)^2}$ at its singularities.*

Solution. *The given function has simple poles at $z = 1$, ω, ω^2 and a pole of order 2 at $z = -1$, ω being the imaginary cube root of unity. Therefore,*

$$\begin{aligned} Res\ (f; 1) &= \lim_{z\to 1}(z - 1)f(z) \\ &= \lim_{z\to 1} \frac{1}{(z + 1)^2(1 + z + z^2)} = \frac{1}{12}. \end{aligned}$$

$$\begin{aligned} Res\ (f;\omega) &= \lim_{z\to\omega}(z-\omega)f(z) \\ &= \lim_{z\to\omega}\frac{1}{(z+1)^2(z-1)(z-\omega^2)} = \frac{1}{3}. \end{aligned}$$

$$\begin{aligned} Res\ (f;\omega^2) &= \lim_{z\to\omega^2}(z-\omega^2)f(z) \\ &= \lim_{z\to\omega^2}\frac{1}{(z+1)^2(z-1)(z-\omega)} = \frac{1}{3}. \end{aligned}$$

$$\begin{aligned} Res\ (f;-1) &= \lim_{z\to -1}\frac{d}{dz}[(z+1)^2 f(z)] \\ &= \lim_{z\to -1}\frac{d}{dz}\left(\frac{1}{z^3-1}\right) \\ &= \lim_{z\to -1}\frac{-3z^2}{(z^3-1)^2} = -\frac{3}{4}. \end{aligned}$$

Example 2. *Find the residue of the function* $f(z) = \cot\ z$ *at* $z=0$.

Solution. *Here* $f(z) = \cot\ z = \frac{\cos z}{\sin z} = \frac{g(z)}{h(z)}$, *say. Clearly* $g(z)$ *is analytic at* $z=0$ *and* $g(0)\neq 0$. *Also* $h(z)$ *is analytic at* $z=0$, $h(0)=0$ *and* $h'(0)\neq 0$. *Therefore,*

$$Res\ (\cot z;0) = \frac{g(0)}{h'(0)} = 1.$$

Example 3. *Find the residue of the function* $F(z) = \frac{z^2+\sin z}{\cos z-1}$ *at its singular points.*

Solution. *Let* $f(z) = z^2+\sin z$ *and* $g(z) = \cos z - 1$. *Now* $g(z) = 0$ *implies* $z = 2n\pi$, $n\in\mathbb{I}$. *Also* $g'(z) = 0$ *at* $z=2n\pi$ *but* $g''(z) = -1$ *at* $z=2n\pi$. *Moreover,* $f(z)$ *has no zero at* $z=2n\pi$. *Also* $f'(2n\pi) = 4n\pi+1$ *and* $g'''(2n\pi) = 0$. *Replacing these values in (2) we obtain*

$$Res(F(z);2n\pi) = Res\left(\frac{z^2+\sin z}{\cos z-1};2n\pi\right) = -2(4n\pi+1),\ n\in\mathbb{I}.$$

Residue at an Essential Singularity

In this case one has to expand the function into Laurent series. For instance, $z=0$ is an essential singularity of the function $f(z) = e^{-1/z}$. The Laurent expansion of $f(z)$ about $z=0$ is

$$1-\frac{1}{z}+\frac{1}{2!z^2}-\frac{1}{3!z^3}+\ \dots\ .$$

Therefore, $Res\ (e^{-1/z};0) = -1$.

Residue at the Point at Infinity

Suppose that $z = \infty$ is an isolated singularity of $f(z)$. Then the residue of $f(z)$ at $z = \infty$ is defined as follows :

$$Res\ (f; \infty)\ = -\frac{1}{2\pi i} \int_C f(z) dz$$

where C is any positively oriented simple closed contour outside of which the function f is analytic and does not have any singularity other than the point at infinity.

Theorem 4. (Cauchy's Residue Theorem)

Suppose that $f(z)$ is analytic inside and on a simple closed contour C except for isolated singularities at $z_1,\ z_2,\ \ldots,\ z_n$ inside C. Then

$$\int_C f(z) dz\ = 2\pi i \sum_{k=1}^{n} Res\ (f; z_k).$$

Proof. Let $C_1,\ C_2,\ \ldots C_n$ be n circles having centers at $z_1,\ z_2,\ \ldots z_n$ and radii so small that they lie entirely within C and do not overlap (see Fig.1). Then f is analytic in the

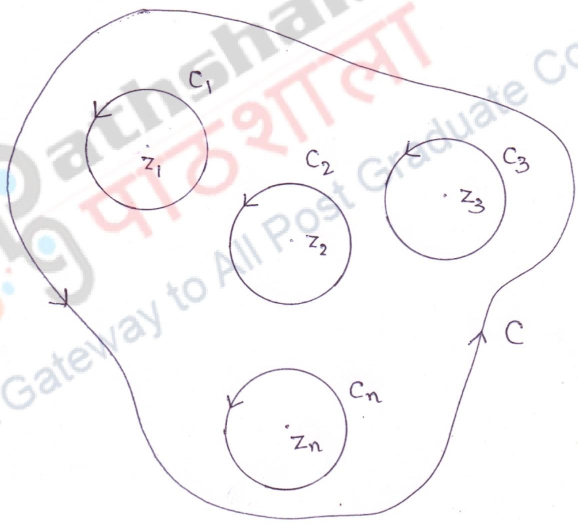

Fig. 1:

region bounded by C and the circles $C_1,\ C_2,\ \ldots C_n$. So by Cauchy's integral formula for multiply connected domains, we have

$$\int_C f(z) dz\ = \sum_{k=1}^{n} \int_{C_k} f(z) dz. \tag{4}$$

Since $Res\ (f; z_k)\ = \frac{1}{2\pi i} \int_{C_k} f(z) dz,\ k\ = 1,\ 2,\ \ldots n$, we obtain from (4) that

$$\int_C f(z) dz\ = 2\pi i \sum_{k=1}^{n} Res\ (f; z_k).$$

This proves the theorem. □

Theorem 5. (Residue Theorem for $\mathbb{C}_\infty$)

Suppose $f(z)$ is analytic in $\mathbb{C}_\infty$ except for isolated singularities at $z_1,\ z_2,\ \ldots,\ z_n,\ \infty$. Then the sum of its residues (including the point at infinity) is zero. That is,

$$Res\ (f;\infty) + \sum_{k=1}^{n} Res\ (f; z_k)\ = 0.$$

Proof. We consider the closed contour C containing all n singularities $z_1,\ z_2,\ \ldots,\ z_n$ located at a finite distance from the point $z = 0$. So by Cauchy's residue theorem

$$\frac{1}{2\pi i}\int_C f(z)dz\ = \sum_{k=1}^{n} Res\ (f; z_k). \tag{5}$$

Also

$$Res\ (f;\infty)\ = -\frac{1}{2\pi i}\int_C f(z)dz. \tag{6}$$

From (5) and (6) we obtain

$$\sum_{k=1}^{n} Res\ (f; z_k)\ = -Res\ (f;\infty)$$

$$i.e.\ \ Res\ (f;\infty) + \sum_{k=1}^{n} Res\ (f; z_k)\ = 0.$$

This proves the result. □

Example 4. *Obtain Cauchy's integral formula from Cauchy's residue theorem.*

Solution. *Suppose that $f(z)$ is analytic in a domain D with boundary γ. Also suppose that f is continuous on γ and z_0 is an arbitrary point inside γ. Let $g(z)\ = \frac{f(z)}{z-z_0}$. Then $g(z)$ has a simple pole at z_0 and hence*

$$Res\ (g; z_0)\ = \lim_{z\to z_0}(z-z_0)g(z)\ = f(z_0).$$

Now applying Cauchy's residue theorem on $g(z)$ we obtain

$$\int_\gamma g(z)dz\ = 2\pi i\cdot f(z_0).$$

This gives

$$f(z_0)\ = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}dz,$$

which is the Cauchy's integral formula.

Example 5. *Evaluate* $\int_{|z|=2} f(z)dz$ *where* $f(z) = \frac{e^z}{z(z-1)^2}$.

Solution. *Let* $C : |z| = 2$. *Clearly* $f(z)$ *has a simple pole at* $z = 0$ *and a pole of order* 2 *at* $z = 1$, *both of which lie inside the curve of integration* C. *Therefore by Cauchy's residue theorem we have*

$$\int_{|z|=2} f(z)dz = \int_{|z|=2} \frac{e^z}{z(z-1)^2}dz = 2\pi i[Res\ (f;0) + Res\ (f;1)].$$

Now

$$Res\ (f;0) = \lim_{z\to 0} zf(z) = \lim_{z\to 0} \frac{e^z}{(z-1)^2} = 1.$$

$$Res\ (f;1) = \lim_{z\to 1} \frac{d}{dz}(z-1)^2 f(z) = \lim_{z\to 1} \frac{d}{dz}\left(\frac{e^z}{z}\right) = \lim_{z\to 1} \frac{ze^z - e^z}{z^2} = 0.$$

Therefore,

$$\int_{|z|=2} \frac{e^z}{z(z-1)^2}dz = 2\pi i[1+0] = 2\pi i.$$

Example 6. *Evaluate* $\int_{|z|=3} \frac{z}{z^4-1}dz$.

Solution. *Let* $f(z) = \frac{z}{z^4-1}$. *Here the curve of integration is* $C : |z| = 3$. *Clearly* $f(z)$ *has simple poles at* $z = \pm 1$ *and at* $z = \pm i$, *all lies inside* C. *Therefore by Cauchy's residue theorem we have*

$$\int_{|z|=3} \frac{z}{z^4-1}dz = 2\pi i[Res\ (f;1) + Res\ (f;-1) + Res\ (f;i) + Res\ (f;-i)].$$

Now

$$Res\ (f;1) = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}(z-1)\frac{z}{z^4-1} = \frac{1}{4}.$$

$$Res\ (f;-1) = \lim_{z\to -1}(z+1)f(z) = \lim_{z\to -1}(z+1)\frac{z}{z^4-1} = \frac{1}{4}.$$

$$Res\ (f;i) = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}(z-i)\frac{z}{z^4-1} = -\frac{1}{4}.$$

$$Res\ (f;-i) = \lim_{z\to -i}(z+i)f(z) = \lim_{z\to -i}(z+i)\frac{z}{z^4-1} = -\frac{1}{4}.$$

Hence

$$\int_{|z|=3} \frac{z}{z^4-1}dz = 0.$$